

ON THE TOPOLOGY OF STABLE MAPS

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ABSTRACT. We investigate how Viro's integral calculus applies for the study of the topology of stable maps. We also discuss several applications to Morin maps and complex maps.

1. Introduction

It is well known that there is a deep relation between the topology of a manifold and the topology of the critical locus of maps. The best example of this fact is Morse Theory which gives the homotopy type of a compact manifold in terms of the Morse indices of the critical points of a Morse function. Let us mention other examples.

R. Thom [15] proved that the Euler characteristic of a compact manifold M of dimension at least 2 had the same parity as the number of cusps of a generic map $f : M \rightarrow \mathbb{R}^2$. Latter H. I. Levine [10] improved this result giving an equality relating $\chi(M)$ and the critical set of f . In [5], T. Fukuda generalized Thom's result to Morin maps $f : M \rightarrow \mathbb{R}^p$ when $\dim M \geq p$. He proved that:

$$(1.1) \quad \chi(M) + \sum_{k=1}^p \chi(\overline{A_k(f)}) = 0 \pmod{2},$$

where $A_k(f)$ is the set of points x in M such that f has a singularity of type A_k at x (see Section 4 for the definition of A_k). Furthermore if f has only fold points (i.e., singularities of type A_1), then T. Fukuda gave an equality relating $\chi(M)$ to the critical set of f . T. Fukuda's formulas were extended to the case of a Morin mapping $f : M \rightarrow N$, where $\dim M \geq \dim N$, by O. Saeki [14]. When $\dim M = \dim N$, similar formulas were obtained by J. R. Quine [13] and I. Nakai [11]. On the other hand, Y. Yomdin [19] showed the equality among Euler characteristics of singular sets of holomorphic maps. As Y. Yomdin and I. Nakai showed in this context, the integral calculus due to O. Viro [18] is useful to find relations like (1.1) for stable maps. In this paper, we investigate how Viro's integral calculus applies in enough wide setup. To do this we introduce the notion of local triviality at infinity and give some examples to illustrate this notion in section 3. T. Ohmoto showed that Yomdin-Nakai's formula is generalized to the statement in terms of characteristic class and discuss a relation with Thom polynomial in his lecture of the conference on the occasion of 70th birthday of T. Fukuda held on 20 July 2010.

We consider a stable map $f : M \rightarrow N$ between two smooth manifolds M and N . We assume that $\dim M \geq \dim N$, that N is connected and that M and N have finite topological types. We also assume that f is locally trivial at infinity (see Definition 3.1) and has finitely many singularity types. Then the singular set $\Sigma(f)$ of f is decomposed into a finite union $\sqcup_{\nu} \nu(f)$, where $\nu(f)$ is the set of singular points of f of type ν . In Theorems 5.1, 5.6, 5.7 and 5.11, we establish several formulas between the Euler characteristics with compact support of M , N and the $\nu(f)$'s. We apply them to maps having singularities of type A_k or D_k in Corollaries 5.4, 5.5, 5.9, 5.10, 5.12 and 5.13.

In Section 6 of this paper, we apply the results of Section 5 to Morin maps and we use the link between the Euler characteristic with compact support and the topological Euler characteristic

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to recover and improve several results of T. Fukuda, T. Fukuda and G. Ishikawa, I. Nakai, J. Quine, O. Saeki. We end the paper with some remarks in the complex case in Section 7.

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2. Viro's integral calculus

In this section, we recall the method of integration with respect to a finitely-additive measure due to O. Viro [17].

Let X be a set and $\mathcal{S}(X)$ denote a collection of subsets of X which satisfies the following properties:

- If $A, B \in \mathcal{S}(X)$, then $A \cup B \in \mathcal{S}(X)$, $A \cap B \in \mathcal{S}(X)$.
- If $A \in \mathcal{S}(X)$, then $X \setminus A \in \mathcal{S}(X)$.

Let R be a commutative ring. Let $\mu_X : \mathcal{S}(X) \rightarrow R$ be a map which satisfies the following properties:

- If A and B are homeomorphic then $\mu_X(A) = \mu_X(B)$.
- For $A, B \in \mathcal{S}(X)$, $\mu_X(A \cup B) = \mu_X(A) + \mu_X(B) - \mu_X(A \cap B)$.

Example 2.1. The Euler characteristic of the homology with compact support, denoted by χ_c , satisfies these conditions for μ_X with $R = \mathbb{Z}$. The mod 2 Euler characteristic also satisfies these conditions for μ_X with $R = \mathbb{Z}/2\mathbb{Z}$.

Let $\text{Cons}(X, \mathcal{S}(X), R)$ (or $\text{Cons}(X)$, for short) denote the set of finite R -linear combinations of characteristic functions $\mathbf{1}_A$ of elements A of $\mathcal{S}(X)$. For $B \in \mathcal{S}(X)$ and $\varphi \in \text{Cons}(X, \mathcal{S}(X), R)$, we define the integral of φ over B with respect to μ_X , denoted by $\int_B \varphi d\mu_X$, by:

$$\int_B \varphi(x) d\mu_X(x) = \sum_A \lambda_A \mu_X(A \cap B) \quad \text{where} \quad \varphi = \sum_A \lambda_A \mathbf{1}_A.$$

We remark that $\mu_X(B) = \int_B d\mu_X$.

Now we are going to state a Fubini type theorem for this integration. We need to introduce some notations.

We say that $(\mathcal{S}(X), \mathcal{S}(Y))$ **fits to** the map $f : X \rightarrow Y$ if the following conditions hold:

- If $A \in \mathcal{S}(X)$, then $f(A) \in \mathcal{S}(Y)$.
- $f^{-1}(y) \in \mathcal{S}(X)$ for $y \in Y$.

- For $A \in \mathcal{S}(X)$, $B \in \mathcal{S}(Y)$ with $f(A) = B$, if $f|_A : A \rightarrow B$ is a locally trivial fibration with fiber F , then:

$$\mu_X(A) = \mu_X(F)\mu_Y(B).$$

- For $A \in \mathcal{S}(X)$, there is a filtration $\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_l = Y$ with $B_i \in \mathcal{S}(Y)$ such that:

$$f|_{f^{-1}(B_i \setminus B_{i-1}) \cap A} : f^{-1}(B_i \setminus B_{i-1}) \cap A \rightarrow B_i \setminus B_{i-1} \quad (i = 0, 1, \dots, l)$$

is a locally trivial fibration.

Lemma 2.2 (Fubini's theorem). *For $\varphi \in \text{Cons}(X)$ and $f : X \rightarrow Y$ such that $(\mathcal{S}(X), \mathcal{S}(Y))$ fits to f , we have:*

$$\int_X \varphi(x) d\mu_X = \int_Y f_* \varphi(y) d\mu_Y \quad \text{where} \quad f_* \varphi(y) = \int_{f^{-1}(y)} \varphi(x) d\mu_X.$$

Proof. It is enough to show the case when $\varphi_X = \mathbf{1}_A$ for $A \in \mathcal{S}(X)$. So let us show that:

$$\mu_X(A) = \int_Y \mu_X(A \cap f^{-1}(y)) d\mu_Y.$$

We take a filtration $\emptyset \subset B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_l$ ($B_i \in \mathcal{S}(Y)$) so that:

$$f|_{f^{-1}(B_i \setminus B_{i-1}) \cap A} : f^{-1}(B_i \setminus B_{i-1}) \cap A \rightarrow B_i \setminus B_{i-1} \quad (i = 0, 2, \dots, l)$$

is a locally trivial fibration with a fiber F_i . Then we have:

$$\begin{aligned} \mu_X(A) &= \sum_{i=0}^l \mu_X(f^{-1}(B_i \setminus B_{i-1}) \cap A) \quad (\text{additivity of } \mu) \\ &= \sum_{i=0}^l \mu_X(F_i) \mu_Y(B_i \setminus B_{i-1}) \quad (\text{triviality of } f|_A \text{ on } B_i \setminus B_{i-1}) \\ &= \sum_{i=0}^l \mu_X(F_i) \int_{B_i \setminus B_{i-1}} d\mu_Y \quad (\text{definition of } \int) \\ &= \sum_{i=0}^l \int_{B_i \setminus B_{i-1}} \mu_X(F_i) d\mu_Y \\ &= \sum_{i=0}^l \int_{B_i \setminus B_{i-1}} \mu_X(A \cap f^{-1}(y)) d\mu_Y \quad (F_i = A \cap f^{-1}(y) \text{ for } y \in B_i \setminus B_{i-1}) \\ &= \int_Y \mu_X(A \cap f^{-1}(y)) d\mu_Y \quad (\text{additivity of } \int). \end{aligned}$$

□

Corollary 2.3. *Set $X_i = \{x \in X \mid \varphi(x) = i\}$, and $Y_j = \{y \in Y \mid f_* \varphi(y) = j\}$. Then we have:*

$$\sum_i i \mu_X(X_i) = \sum_j j \mu_Y(Y_j).$$

Proof. This is clear, since:

$$\begin{aligned} \int_X \varphi(x) d\mu_X &= \sum_i \int_{X_i} \varphi(x) d\mu_X = \sum_i \int_{X_i} i d\mu_X = \sum_i i \mu_X(X_i), \\ \int_Y f_* \varphi(y) d\mu_Y &= \sum_j \int_{Y_j} f_* \varphi(y) d\mu_Y = \sum_j \int_{Y_j} j d\mu_Y = \sum_j j \mu_Y(Y_j). \end{aligned}$$

□

Corollary 2.4. *If $f_* \varphi$ is a constant d on $y \in Y$, we have:*

$$\sum_i i \mu_X(X_i) = d \mu_Y(Y).$$

In the sequel, we will apply O. Viro's integral calculus to investigate topology of stable maps (see [11] and [12] for a similar strategy).

3. Local triviality at infinity

In this section, we define the notion of local triviality at infinity for a smooth map.

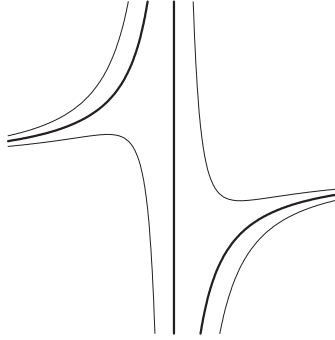
Definition 3.1. Let $f : M \rightarrow N$ be a smooth map between two smooth manifolds. We say f is **locally trivial at infinity at** $y \in N$ if there are a compact set K in M and an open neighborhood D of y such that $f : (M \setminus K) \cap f^{-1}(D) \rightarrow D$ is a trivial fibration. We say f is **locally trivial at infinity** if it is locally trivial at infinity at any $y \in N$.

Here are some examples of functions not locally trivial at infinity.

Example 3.2 (Broughton [2]). Consider $f(x, y) = x(xy + 1)$. The critical set $\Sigma(f)$ of f is empty. For $t \neq 0$,

$$f^{-1}(t) = \{y = (t - x)/x^2\}.$$

We have $f^{-1}(t) = \mathbb{R}^*$, $f^{-1}(0) = \mathbb{R} \cup \mathbb{R}^*$ and $\chi_c(f^{-1}(t)) = -2$, $\chi_c(f^{-1}(0)) = -3$. So this example is not locally trivial at infinity on $t = 0$. The level curves of f with level $-1/2, 0, 1/2$ are shown in the figure. The thick line shows the level 0.

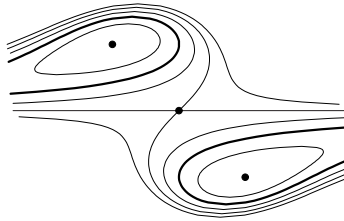


A map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\Sigma(f) = \emptyset$ may not be surjective. M. Shiota remarked that the map $\mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto (x(xy + 1) + 1)^2 + x^2$, has empty critical set, and is not surjective.

Example 3.3 (Tibăr-Zaharia [16, Example 3.2]). Consider $f(x, y) = x^2y^2 + 2xy + (y^2 - 1)^2$. Then $\Sigma(f) = \{(0, 0), (1, -1), (-1, 1)\}$ and $f(0, 0) = 1$, $f(1, -1) = f(-1, 1) = -1$. Since $f^{-1}(t)$ is two lines (resp. circles) if $0 \leq t < 1$ (resp. $-1 < t < 0$), we have:

$$\chi_c(f^{-1}(t)) = \begin{cases} -2 & (0 \leq t < 1) \\ 0 & (-1 < t < 0). \end{cases}$$

So this example is not locally trivial at infinity on $t = 0$. The level curves of f with level $-1, -1/2, 0, 1/2, 1, 3/2$ are shown in the figure. The thick line shows the level 0.



4. Euler characteristics of local generic fibers

In this section, we present a general method for the computations of the Euler characteristic of the Milnor fibers of a stable map-germ. We start with a lemma.

Lemma 4.1. *Let Y be a manifold and let X be a set defined by:*

$$X = \{(x, y) \in \mathbb{R}^p \times Y : x_1^2 + \cdots + x_p^2 = g(y)\}$$

where $g(y)$ is a smooth positive function. Then X is a smooth manifold and:

$$\chi_c(X) = \chi(S^{p-1})\chi_c(Y) = (1 - (-1)^p)\chi_c(Y).$$

Proof. It is easy to check that X is a manifold. To obtain the equality, consider the map:

$$X \rightarrow Y, (x, y) \mapsto y.$$

This is a locally trivial fibration whose fiber is S^{p-1} . □

Example 4.2. Let X be the set defined by:

$$X = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x_1^2 + \cdots + x_p^2 = y_1^2 + \cdots + y_q^2 + 1\}.$$

Since $X \rightarrow \mathbb{R}^q, (x, y) \mapsto y$, is a locally trivial fibration whose fiber is S^{p-1} , we have:

$$\chi_c(X) = \chi_c(S^{p-1})\chi_c(\mathbb{R}^q) = (1 - (-1)^p)(-1)^q = (-1)^q - (-1)^{p+q}.$$

Example 4.3. Let X be the set defined by:

$$X = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x_1^2 + \cdots + x_p^2 = y_1^2 + \cdots + y_q^2\}.$$

Since $X \setminus \{0\} \rightarrow \mathbb{R}^q \setminus \{0\}, (x, y) \mapsto y$, is a locally trivial fibration whose fiber is S^{p-1} , we have:

$$\begin{aligned} \chi_c(X) &= \chi_c(\{0\}) + \chi_c(S^{p-1})\chi_c(\mathbb{R}^q \setminus \{0\}) \\ &= 1 + (1 - (-1)^p)((-1)^q - 1) \\ &= (-1)^p + (-1)^q - (-1)^{p+q}. \end{aligned}$$

Next we will apply this lemma and these examples to the computation of Euler characteristics of local nearby fibers of stable map-germs. The general setting is the following. Let \tilde{B} be a small open ball in \mathbb{R}^n centered at 0 and let B' be a small open ball in \mathbb{R}^{a+b} centered at 0. We consider a map f defined by:

$$(4.1) \quad f : \tilde{B} \times B' \times \mathbb{R}^h \rightarrow \mathbb{R} \times \mathbb{R}^j \times \mathbb{R}^h, \quad (x, z, c) \mapsto (g(x; c) + Q(z), g'(x; c), c)$$

where $Q(z) = z_1^2 + \cdots + z_a^2 - z_{a+1}^2 - \cdots - z_{a+b}^2$. Remember that stable-germs are versal unfoldings, deleting constant terms, of a map-germ $x \mapsto (g(x; 0), g'(x; 0))$, called the genotype, and can be written in this form. (See [1, Part I, 9].)

We want to compute the Euler characteristic of a local generic fiber around the point $(0, 0, 0)$, namely the fiber $f^{-1}(\varepsilon, \varepsilon', c)$ for small ε and ε' . First we remark that $f^{-1}(\varepsilon, \varepsilon', c)$ is diffeomorphic to:

$$F = \{(x, z) \in B \times B' : g(x; c) + Q(z) = \varepsilon\},$$

where B is the nonsingular subset of \tilde{B} defined by $g'(x; c) = \varepsilon'$. Note that $\dim F = n - j + a + b - 1$ and $\dim B = n - j$.

Lemma 4.4. *We have:*

$$\chi_c(F) = \begin{cases} \chi_c(B_0) & a \text{ even}, b \text{ even} \\ \chi_c(B) + \chi_c(B_+) - \chi_c(B_-) & a \text{ even}, b \text{ odd} \\ \chi_c(B) - \chi_c(B_+) + \chi_c(B_-) & a \text{ odd}, b \text{ even} \\ -2\chi_c(B) - \chi_c(B_0) & a \text{ odd}, b \text{ odd} \end{cases}$$

where:

$$\begin{aligned} B_+ &= \{x \in B \mid g(x; c) > \varepsilon\}, \\ B_0 &= \{x \in B \mid g(x; c) = \varepsilon\}, \\ B_- &= \{x \in B \mid g(x; c) < \varepsilon\}. \end{aligned}$$

Remark that B , B_+ , B_- and B_0 depend on ε , ε' , c and it would be better to denote them by $B(\varepsilon, \varepsilon', c)$, $B_+(\varepsilon, \varepsilon', c)$, $B_-(\varepsilon, \varepsilon', c)$ and $B_0(\varepsilon, \varepsilon', c)$ respectively. But we keep the notation in the lemma for shortness.

Proof. Consider the map: $\varphi : F \rightarrow B$, $(y, z) \mapsto y$. The singular set of φ is described by:

$$\text{rank} \begin{pmatrix} g_{y_i} & Q_{z_k} \\ 1 & 0 \end{pmatrix} < m + 1 \quad (\text{i.e., } Q_{z_1} = \cdots = Q_{z_{a+b}} = 0),$$

where $m = n - j$ and (y_1, \dots, y_m) denotes a local coordinates system for B . Note that, with the standard notation, $\Sigma(\varphi) = \Sigma^{a+b}(\varphi)$. Now we consider the singular set of $\varphi|_{\Sigma(\varphi)}$, which is defined by:

$$\text{rank} \begin{pmatrix} g_{y_i} & Q_{z_k} \\ 1 & 0 \\ 0 & Q_{z_i z_j} \end{pmatrix} < m + a + b.$$

Since Q is quadratic, we have $\Sigma^{a+b,1}(\varphi) = \emptyset$ which means that φ is a fold map. Moreover $\Sigma(\varphi)$ is included in $\varphi^{-1}(B_0)$. Hence $\varphi|_{\varphi^{-1}(B_+)}$ and $\varphi|_{\varphi^{-1}(B_-)}$ are locally trivial. Furthermore the decomposition $\varphi^{-1}(B_0) = \Sigma(\varphi) \sqcup (\varphi^{-1}(B_0) \setminus \Sigma(\varphi))$ gives a Whitney stratification of $\varphi^{-1}(B_0)$ and $\varphi|_{\Sigma(\varphi)}$ and $\varphi|_{\varphi^{-1}(B_0) \setminus \Sigma(\varphi)}$ have no critical point so $\varphi|_{\varphi^{-1}(B_0)}$ is also trivial by the Thom-Mather lemma.

Using Examples 4.2 and 4.3, we remark the following:

$$\chi_c(\varphi^{-1}(x)) = \begin{cases} (-1)^b - (-1)^{a+b} & x \in B_- \\ (-1)^a - (-1)^{a+b} & x \in B_+ \\ (-1)^a + (-1)^b - (-1)^{a+b} & x \in B_0 \end{cases}$$

In other words, $\chi_c(\varphi^{-1}(x))$ is given by the following table:

	$x \in B_+$	$x \in B_-$	$x \in B_0$
a even, b even	0	0	1
a even, b odd	2	0	1
a odd, b even	0	2	1
a odd, b odd	-2	-2	-3

Therefore, using the local trivialities mentioned above, we conclude as follows:

$$\chi_c(F) = \begin{cases} \chi_c(B_0) & a \text{ even, } b \text{ even} \\ 2\chi_c(B_+) + \chi_c(B_0) = \chi_c(B) + \chi_c(B_+) - \chi_c(B_-) & a \text{ even, } b \text{ odd} \\ 2\chi_c(B_-) + \chi_c(B_0) = \chi_c(B) - \chi_c(B_+) + \chi_c(B_-) & a \text{ odd, } b \text{ even} \\ -2\chi_c(B_+) - 2\chi_c(B_-) - 3\chi_c(B_0) = -2\chi_c(B) - \chi_c(B_0) & a \text{ odd, } b \text{ odd} \end{cases}$$

Here we use the fact $\chi_c(B_+) + \chi_c(B_-) + \chi_c(B_0) = \chi_c(B)$. □

If B is an open m -ball, then $\chi_c(B) = (-1)^m$ and we conclude that:

$$1 + (-1)^{m+a+b} \chi_c(F) = \begin{cases} (-1)^m((-1)^m + \chi_c(B_0)) & a \text{ even, } b \text{ even} \\ -(-1)^m(\chi_c(B_+) - \chi_c(B_-)) & a \text{ even, } b \text{ odd} \\ (-1)^m(\chi_c(B_+) - \chi_c(B_-)) & a \text{ odd, } b \text{ even} \\ -(-1)^m((-1)^m + \chi_c(B_0)) & a \text{ odd, } b \text{ odd} \end{cases}$$

If f is an unfolding of a function-germ (i.e., $m = n$, $j = 0$), then B is an open m -ball.

Definition 4.5. We consider an unfolding of a function-germ $(x, z) \mapsto g(x; 0) + Q(z)$. Let σ denote the singularity type of the map $x \mapsto g(x, 0)$. When $m + a + b$ is even, define s_σ by:

$$s_\sigma = 1 + \chi_c(F) = \begin{cases} -\chi_c(B_+) + \chi_c(B_-) & \text{if } m \text{ is odd and } a + b \text{ is odd} \\ 1 + \chi_c(B_0) & \text{if } m \text{ is even and } a + b \text{ is even} \end{cases}$$

When $m + a + b$ is odd, define s_σ^{\max} , s_σ^{\min} by:

$$\begin{aligned} s_\sigma^{\max} &= 1 - \max\{\chi_c(F)\} = \begin{cases} -\max\{-1 + \chi_c(B_0)\} & \text{if } m \text{ is odd and } a + b \text{ is even} \\ \min\{\chi_c(B_+) - \chi_c(B_-)\} & \text{if } m \text{ is even and } a + b \text{ is odd} \end{cases} \\ s_\sigma^{\min} &= 1 - \min\{\chi_c(F)\} = \begin{cases} -\min\{-1 + \chi_c(B_0)\} & \text{if } m \text{ is odd and } a + b \text{ is even} \\ \max\{\chi_c(B_+) - \chi_c(B_-)\} & \text{if } m \text{ is even and } a + b \text{ is odd} \end{cases} \end{aligned}$$

Now let us apply this machinery to A_k and D_k singularities.

4.1. **A_k singularities.** We set $n = m = 1$, $j = 0$ and:

$$g_c(x) = g(x; c) = x^{k+1} + c_1 x^{k-1} + \cdots + c_{k-2} x^2 + c_{k-1} x.$$

Then we have $\chi_c(B_0) = \#\{x : g_c(x) = \varepsilon\}$ and:

$$\chi_c(B_+) - \chi_c(B_-) = \begin{cases} 0 & k \text{ even,} \\ -1 & k \text{ odd.} \end{cases}$$

If k is even, then we obtain:

$$1 - (-1)^{a+b} \chi_c(F) = \begin{cases} 1 - \#\{x \in B : g_c(x) = \varepsilon\} & a \text{ even, } b \text{ even} \\ 0 & a \text{ even, } b \text{ odd} \\ 0 & a \text{ odd, } b \text{ even} \\ \#\{x \in B : g_c(x) = \varepsilon\} - 1 & a \text{ odd, } b \text{ odd} \end{cases}$$

If k is odd, then we obtain:

$$1 - (-1)^{a+b} \chi_c(F) = \begin{cases} 1 - \#\{x \in B : g_c(x) = \varepsilon\} & a \text{ even, } b \text{ even} \\ -1 & a \text{ even, } b \text{ odd} \\ 1 & a \text{ odd, } b \text{ even} \\ \#\{x \in B : g_c(x) = \varepsilon\} - 1 & a \text{ odd, } b \text{ odd} \end{cases}$$

4.2. **Unfoldings of functions** $(x_1, x_2, z) \mapsto g(x, 0) + Q(z)$. We set $n = m = 2$ and $j = 0$. We consider the map defined by:

$$(4.2) \quad (\mathbb{R}^{2+a+b+h}, 0) \rightarrow (\mathbb{R}^{1+h}, 0), \quad (x_1, x_2, z_1, \dots, z_{a+b}, c_1, \dots, c_h) \mapsto (g(x_1, x_2, c_1, \dots, c_h) + z_1^2 + \cdots + z_a^2 - z_{a+1}^2 - \cdots - z_{a+b}^2, c_1, \dots, c_h).$$

Let r denote the number of branches of the curve defined by $g(x; 0) = 0$. Since $\chi_c(B_0) = -r$, we obtain that:

$$1 + (-1)^{a+b} \chi_c(F) = \begin{cases} 1 - r & a \text{ even, } b \text{ even} \\ \chi_c(B_-) - \chi_c(B_+) & a \text{ even, } b \text{ odd} \\ \chi_c(B_+) - \chi_c(B_-) & a \text{ odd, } b \text{ even} \\ r - 1 & a \text{ odd, } b \text{ odd} \end{cases}$$

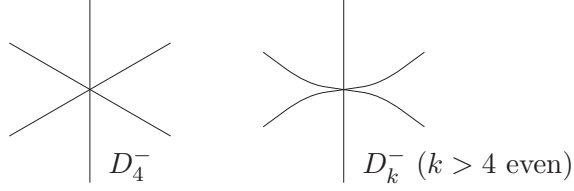
O. Viro [18] described the list of possible smoothings of D_k ($k \geq 4$), E_6 , E_7 , E_8 , J_{10} and non-degenerate r -fold points. In next subsection, we use this list to compute $\chi_c(B_+) - \chi_c(B_-)$ for D_k singularities. We leave to the reader the computation in the other cases.

4.3. D_k singularities. We denote by D_k^\pm the singularity defined by (4.2) with:

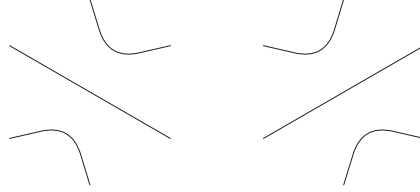
$$g(x; c) = x_1(x_1^{k-2} \pm x_2^2) + c_1x_1 + \cdots + c_{k-2}x_1^{k-2} + c_{k-1}x_2.$$

First case: k is even and $\{x \in \mathbb{R}^2 : g(x, 0) = 0\}$ has 3 branches.

The zero set of $g(x, 0)$ looks like the following:

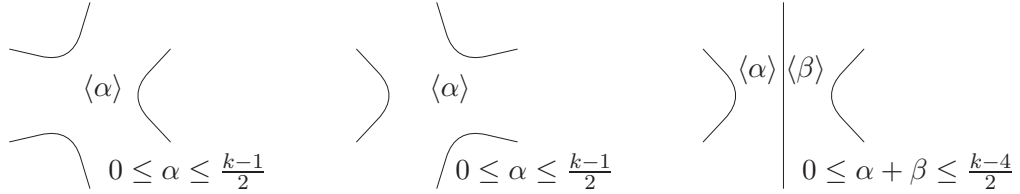


First consider the smoothing described by the following picture:



For such a smoothing, it is easy to see $\chi_c(B_+) - \chi_c(B_-) = 0$.

Next we consider the smoothings described by the following pictures:

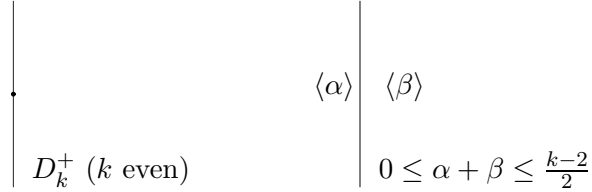


Here $\langle \alpha \rangle$ represents a group of α ovals without nests. For such smoothings, we see that: $\chi_c(B_+) - \chi_c(B_-) = 2(1 + \alpha)$, $-2(1 + \alpha)$, $2(\alpha - \beta)$ respectively. Then we obtain:

$$\chi_c(B_+) - \chi_c(B_-) = -k, -k + 2, \dots, k - 2, k.$$

Second case: k is even and $\{x \in \mathbb{R}^2 : g(x, 0) = 0\}$ has 1 branch.

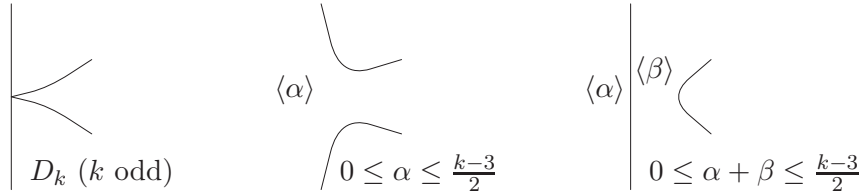
The smoothings are described by the figure on the right-hand side.



For such smoothings, we see that $\chi_c(B_+) - \chi_c(B_-) = 2(\alpha - \beta)$. Thus we have:

$$\chi_c(B_+) - \chi_c(B_-) = 2 - k, 4 - k, \dots, k - 4, k - 2.$$

Third case: k is odd.



For such smoothings, we see that: $\chi_c(B_+) - \chi_c(B_-) = -1 - 2\alpha$, $1 - 2(\alpha - \beta)$, respectively. Thus we have:

$$\chi_c(B_+) - \chi_c(B_-) = 2 - k, 4 - k, \dots, k - 4, k - 2.$$

5. Study of stable maps $f : M \rightarrow N$ with $\dim M \geq \dim N$

Let $f : M \rightarrow N$ be a stable map between two smooth manifolds M and N . Let $m = \dim M$ and $n = \dim N$. We assume that $m \geq n$, that N is connected and that M and N have finite topological types. Let σ denote the singularity type given by the genotype : $x \mapsto (g(x; 0), g'(x; 0))$ in the notation of (4.1). Then the genotype σ gives rise to two singularity types of f : we say that f is of type σ^+ (resp. σ^-) if, in the expression of given (4.1), b is even (resp. odd). The definition of σ^+ and σ^- is ad hoc, since it depends on the normal form (4.1). It seems to be no natural way to define the sign in general. We set:

$$\sigma^\pm(f) = \{x \in M : f_x \text{ has singularity of type } \sigma^\pm\},$$

where $f_x : (M, x) \rightarrow (N, f(x))$ is the germ of f at x . Let $\Sigma(f)$ denote the critical set of f .

Since f is stable, $\Sigma(f) \cap f^{-1}(y)$ is a finite set for each $y \in N$. Then f defines a multi-germ:

$$f_y : (M, \Sigma(f) \cap f^{-1}(y)) \rightarrow (N, y).$$

Let τ denote a type of singularities of stable multi-germs and:

$$N_\tau(f) = \{y \in N : f_y \text{ has singularities of type } \tau\}.$$

5.1. Case $m - n$ is odd. If $m - n$ is odd, then $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$ does not depend on the choice of regular value y' nearby $f(x)$, where $B_\varepsilon(x)$ denotes the open ball of small radius ε centered at x in M . Indeed, $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is a compact odd-dimensional manifold with boundary and so:

$$\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \frac{1}{2} \chi(f^{-1}(y') \cap \partial \overline{B_\varepsilon(x)}).$$

But the last Euler characteristic is equal to $\chi(f^{-1}(f(x)) \cap \partial \overline{B_\varepsilon(x)})$. If x is of type ν then we denote by c_ν the Euler characteristic $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$.

Replacing the ball of small radius with a ball with big radius and assuming that f is locally trivial at infinity, we may establish in a similar way that $\chi_c(f^{-1}(y))$ does not depend on the choice of the regular value y of f . We denote this Euler characteristic by χ_f .

Theorem 5.1. *Assume that $f : M \rightarrow N$ is locally trivial at infinity and has finitely many singularity types (this is the case when (m, n) is a pair of nice dimensions in Mather's sense). Then we have:*

$$(5.1) \quad \sum_{\nu} c_{\nu} \chi_c(\nu(f)) = \chi_f \chi_c(N),$$

provided that the $\chi_c(\nu(f))$'s and χ_f are finite. Moreover, if all singularities of f are versal unfoldings of function-germs then we have:

$$(5.2) \quad \chi_c(M) - \chi_f \chi_c(N) = \sum_{\sigma} s_{\sigma} [\chi_c(\sigma^+(f)) - \chi_c(\sigma^-(f))],$$

where σ denotes the singularity type of the genotype and s_{σ} is defined as in Definition 4.5.

Proof. We consider the stratification of f defined by the types of singularities (see Nakai's paper [12, §1]) and we define $\mathcal{S}(M)$, $\mathcal{S}(N)$ as the subset algebras generated by the strata and fibers of f . Then $(\mathcal{S}(X), \mathcal{S}(Y))$ fits to the map f . Set $\mu_X = \chi_c$, $\mu_Y = \chi_c$ and:

$$\varphi(x) = \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}),$$

where y' is a regular value nearby $f(x)$. Applying Corollary 2.3 for φ , Lemma 5.2 and Remark 5.3 below, we obtain:

$$(5.3) \quad \sum_{\nu} c_{\nu} \chi_c(\nu(f)) = \chi_f \chi_c(N).$$

By the additivity of the Euler characteristic with compact support, we get:

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_{\nu} (1 - c_{\nu}) \chi_c(\nu(f)).$$

If all the singularities are versal unfoldings of function-germs then each genotype gives rise to two singularity types $\sigma^+(f)$ and $\sigma^-(f)$ and $1 - c_{\sigma^-} = -(1 - c_{\sigma^+}) = -s_\sigma$ by Remark 5.3 below and the computations made in Section 4. \square

Lemma 5.2. *Let $f : M \rightarrow N$ be a smooth map such that:*

- $\dim M - \dim N$ is odd,
- $f|_{\Sigma(f)}$ is finite,
- f is locally trivial at infinity.

Then for each $y \in N$, we have:

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi(x) d\chi_c = \chi_c(f^{-1}(y')),$$

where y' is a regular value of f close to y .

Proof. Set $\{x_1, \dots, x_s\} = f^{-1}(y) \cap \Sigma(f)$. Take a regular value y' of f near y . Then:

$$\begin{aligned} \chi_c(f^{-1}(y')) &= \chi_c(f^{-1}(y') \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &= \chi_c(f^{-1}(y) \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &= \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi(x_i) \\ &= \int_{f^{-1}(y) \setminus \{x_1, \dots, x_s\}} \varphi(x) d\chi_c + \int_{\{x_1, \dots, x_s\}} \varphi(x) d\chi_c \\ &= \int_{f^{-1}(y)} \varphi(x) d\chi_c. \end{aligned}$$

\square

Remark 5.3. Set $\phi(x) = \chi_c(f^{-1}(y') \cap B_\varepsilon(x))$ where y' is a regular value nearby $f(x)$. Then:

$$\varphi(x) = \phi(x) + \chi_c(f^{-1}(y') \cap S_\varepsilon(x)),$$

where $S_\varepsilon(x)$ is the sphere of radius ε centered at x . If $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is an odd dimensional manifold with boundary $f^{-1}(y') \cap S_\varepsilon(x)$, then we obtain $\phi(x) = -\varphi(x)$, since:

$$2\varphi(x) = \chi_c(f^{-1}(y') \cap S_\varepsilon(x)) = -2\phi(x).$$

Similarly if $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is an even dimensional manifold with boundary $f^{-1}(y') \cap S_\varepsilon(x)$, we obtain that $\phi(x) = \varphi(x)$.

Corollary 5.4. *Assume that the map f satisfies the assumptions of Theorem 5.1 and has at worst A_n singularities. Then, we have:*

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_{k: \text{odd}} [\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))].$$

Proof. Using the computations in Section 4, we see that $s_{A_k} = 0$ if k is even and $s_{A_k} = 1$ if k is odd. \square

Corollary 5.5. *Assume that the map f satisfies the assumptions of Theorem 5.1 and has only stable singularities locally defined by (4.2). We denote σ_r the union of singularities types so that the number of branches of $g(x_1, x_2; 0) = 0$ near 0 is r . We denote by σ_r^+ (resp. σ_r^-) the union of such singularities types with even (resp. odd) b . Then, we have:*

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_r (1 - r) [\chi_c(\sigma_r^+(f)) - \chi_c(\sigma_r^-(f))].$$

Proof. Using the computations in Section 4, we see that $s_{\sigma_r} = 1 - r$. \square

5.2. Case $m - n$ is even and $m - n > 0$. If $m - n$ is even and non-zero then $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$ depends on the choice of the regular value y' nearby $f(x)$. But its parity does not depend on y' . Indeed, $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is a compact even-dimensional manifold with boundary and so:

$$\begin{aligned}\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) &\equiv \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) \\ &\equiv \psi(f^{-1}(y') \cap S_\varepsilon(x)) \\ &\equiv \psi(f^{-1}(f(x)) \cap S_\varepsilon(x)) \pmod{2},\end{aligned}$$

where ψ denotes the semi-characteristic, i.e., half the sum of the mod 2 Betti numbers (see [20]).

If a point x in M is of singular type ν then we denote by c_ν the mod 2 Euler characteristic $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$. We will denote by χ_f the mod 2 Euler characteristic $\chi_c(f^{-1}(y))$ where y is a regular value of f . The following theorem is proved in the same way as Theorem 5.1.

Theorem 5.6. *Assume that $f : M \rightarrow N$ is locally trivial at infinity and has finitely many singularity types (this is the case when (m, n) is a pair of nice dimensions in Mather's sense). Then we have:*

$$(5.4) \quad \sum_{\nu} c_\nu \chi_c(\nu(f)) \equiv \chi_f \chi_c(N) \pmod{2},$$

provided that the $\chi_c(\nu(f))$'s and χ_f are finite. Moreover, if all singularities of f are versal unfoldings of function-germs then we have:

$$(5.5) \quad \chi_c(M) - \chi_f \chi_c(N) \equiv \sum_{\sigma} s_{\sigma} [\chi_c(\sigma^+(f)) - \chi_c(\sigma^-(f))] \pmod{2},$$

where σ denotes the singularity type of the genotype and s_{σ} is defined as in Definition 4.5.

This theorem gives a mod 2 equality. Nevertheless, it is still possible to find integral relations between the topology of the source, the target and the singular set.

Let ν denote a singularity type of a map-germ. Let c_{ν}^{\max} (resp. c_{ν}^{\min}) denote the maximal (resp. minimum) of all possible Euler characteristics of local regular fibers nearby the singular fiber. Set also:

$$\begin{aligned}N_j^{\max} &= \{y \in N : j = \max\{\chi_c(f^{-1}(y')) : y' \text{ a regular value nearby } y\}\}, \\ N_j^{\min} &= \{y \in N : j = \min\{\chi_c(f^{-1}(y')) : y' \text{ a regular value nearby } y\}\}.\end{aligned}$$

Theorem 5.7. *If a smooth map $f : M \rightarrow N$ is locally trivial at infinity and has finitely many singularity types (this is the case if (m, n) is a pair of nice dimensions in Mather's sense), then:*

$$\begin{aligned}\sum_{\nu} c_{\nu}^{\max} \cdot \chi_c(\nu(f)) &\geq \sum_j j \chi_c(N_j^{\max}), \\ \sum_{\nu} c_{\nu}^{\min} \cdot \chi_c(\nu(f)) &\leq \sum_j j \chi_c(N_j^{\min}),\end{aligned}$$

provided the $\chi_c(\nu(f))$'s, the $\chi_c(N_j^{\max})$'s and the $\chi_c(N_j^{\min})$'s are finite. If f is stable, we have the equalities. Moreover, if all singularities are versal unfoldings of function-germs then:

$$\begin{aligned}\chi_c(M) - \sum_j j \chi_c(N_j^{\max}) &= \sum_{\sigma} [s_{\sigma}^{\max} \chi_c(\sigma^+(f)) - s_{\sigma}^{\min} \chi_c(\sigma^-(f))], \\ \chi_c(M) - \sum_j j \chi_c(N_j^{\min}) &= \sum_{\sigma} [s_{\sigma}^{\min} \chi_c(\sigma^+(f)) - s_{\sigma}^{\max} \chi_c(\sigma^-(f))],\end{aligned}$$

where σ denotes the singularity type of the genotype and s_{σ}^{\max} and s_{σ}^{\min} are defined in Definition 4.5.

Proof. To get the first inequalities, we apply the same method as we did in the proof of Theorem 5.1 with the two following constructible functions φ_{\max} and φ_{\min} :

$$\varphi_{\max}(x) = \max\{\chi_c(f^{-1}(y') \cap B_\varepsilon(x)) : y' \text{ is regular value nearby } f(x)\},$$

$$\varphi_{\min}(x) = \min\{\chi_c(f^{-1}(y')) \cap B_\varepsilon(x) : y' \text{ is regular value nearby } f(x)\}.$$

We also use Lemma 5.8 below.

When f is stable, by the additivity of the Euler characteristic with compact support, we get:

$$\begin{aligned}\chi_c(M) - \sum_j \chi_c(N_j^{\max}) &= \sum_\nu (1 - c_\nu^{\max}) \chi_c(\nu(f)), \\ \chi_c(M) - \sum_j \chi_c(N_j^{\min}) &= \sum_\nu (1 - c_\nu^{\min}) \chi_c(\nu(f)).\end{aligned}$$

If all the singularities are versal unfoldings of function-germs then each genotype gives rise to two singularity types $\sigma_+(f)$ and $\sigma_-(f)$. Using the computations done in Section 4, we see that:

$$1 - c_{\sigma_+}^{\max}(f) = s_\sigma^{\max}, \quad 1 - c_{\sigma_-}^{\max}(f) = -s_\sigma^{\min}, \quad 1 - c_{\sigma_+}^{\min}(f) = s_\sigma^{\min}, \quad \text{and} \quad 1 - c_{\sigma_-}^{\min}(f) = -s_\sigma^{\max}.$$

□

Lemma 5.8. *Let $f : M \rightarrow N$ be a smooth map such that:*

- $\dim M - \dim N$ is even,
- $f|_{\Sigma(f)}$ is finite,
- f is locally trivial at infinity.

Then we have:

$$\begin{aligned}f_*\varphi_{\max}(y) &\geq \max\{\chi_c(f^{-1}(y')) : y' \text{ a regular value nearby } y\}, \\ f_*\varphi_{\min}(y) &\leq \min\{\chi_c(f^{-1}(y')) : y' \text{ a regular value nearby } y\}.\end{aligned}$$

We have the equalities when f is stable.

Proof. Set $\{x_1, \dots, x_s\} = f^{-1}(y) \cap \Sigma(f)$. Take a regular value y' of f near y . Then:

$$\begin{aligned}\chi_c(f^{-1}(y')) &= \chi_c(f^{-1}(y') \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &= \chi_c(f^{-1}(y) \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &\leq \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi_{\max}(x_i) \\ &= \int_{f^{-1}(y) \setminus \{x_1, \dots, x_s\}} \varphi_{\max}(x) d\chi_c + \int_{\{x_1, \dots, x_s\}} \varphi_{\max}(x) d\chi_c \\ &= \int_{f^{-1}(y)} \varphi_{\max}(x) d\chi_c = f_*\varphi_{\max}(y).\end{aligned}$$

When f is stable, we see that the equality is attained by some y' using the fact (i) \iff (iii) of [21, Lemma 1.5].

Similarly we obtain:

$$\begin{aligned}\chi_c(f^{-1}(y')) &= \chi_c(f^{-1}(y') \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &= \chi_c(f^{-1}(y) \setminus \cup_i \overline{B_\varepsilon(x_i)}) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\ &\geq \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi_{\min}(x_i) \\ &= \int_{f^{-1}(y) \setminus \{x_1, \dots, x_s\}} \varphi_{\min}(x) d\chi_c + \int_{\{x_1, \dots, x_s\}} \varphi_{\min}(x) d\chi_c \\ &= \int_{f^{-1}(y)} \varphi_{\min}(x) d\chi_c = f_*\varphi_{\min}(y).\end{aligned}$$

When f is stable, we see that the equality is attained by some y' using the fact (i) \iff (iii) of [21, Lemma 1.5]. \square

Now let us apply this theorem to the case of a map having at worst D_n singularities. Using the computations in Section 4, we see that:

$$1 - c_\nu^{\max} = \begin{cases} -k & \text{if } x \in A_k^+(f), \\ -1 & \text{if } x \in A_k^-(f), \text{ } k \text{ odd}, \\ 0 & \text{if } x \in A_k^-(f), \text{ } k \text{ even}, \\ k & \text{if } x \in D_k^-(f), \text{ } k \text{ even}, \\ k-2 & \text{if } x \in D_k^+(f), \text{ } k \text{ even}, \\ k-2 & \text{if } x \in D_k(f), \text{ } k \text{ odd}, \end{cases} \quad 1 - c_\nu^{\min} = \begin{cases} k & \text{if } x \in A_k^-(f) \\ 1 & \text{if } x \in A_k^+(f), \text{ } k \text{ odd}, \\ 0 & \text{if } x \in A_k^+(f), \text{ } k \text{ even}, \\ -k & \text{if } x \in D_k^-(f), \text{ } k \text{ even}, \\ 2-k & \text{if } x \in D_k^+(f), \text{ } k \text{ even}, \\ 2-k & \text{if } x \in D_k(f), \text{ } k \text{ odd}. \end{cases}$$

Corollary 5.9. *If the map f satisfies the assumptions of Theorem 5.7 and has at worst D_n singularities then:*

$$\begin{aligned} \chi_c(M) - \sum_j j\chi_c(N_j^{\max}) &= - \sum_{k>0} k\chi_c(A_k^+(f)) - \sum_{k:\text{odd}} \chi_c(A_k^-(f)) + \sum_k (k-2)\chi_c(D_k(f)) + 2 \sum_{k:\text{even}} \chi_c(D_k^-(f)) \\ \chi_c(M) - \sum_j j\chi_c(N_j^{\min}) &= \sum_{k>0} k\chi_c(A_k^-(f)) + \sum_{k:\text{odd}} \chi_c(A_k^+(f)) + \sum_k (2-k)\chi_c(D_k(f)) - 2 \sum_{k:\text{even}} \chi_c(D_k^-(f)) \end{aligned}$$

Proof. Combine the previous theorem with the above expressions of φ_{\max} and φ_{\min} . \square

Corollary 5.10. *Assume that a map f satisfies the assumptions of Theorem 5.7 and has at worst A_n singularities.*

When $\dim N = 1$, we have:

$$\sum_j j\chi_c(N_j^{\max}) = \chi_c(M) + \chi_c(A_1(f)), \quad \sum_j j\chi_c(N_j^{\min}) = \chi_c(M) - \chi_c(A_1(f)),$$

and thus:

$$\begin{aligned} \sum_j \frac{j}{2} [\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})] &= \chi_c(M), \\ \sum_j \frac{j}{2} [\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})] &= \chi_c(A_1(f)) = \chi_c(\Sigma(f)). \end{aligned}$$

When $\dim N = 2$, we have:

$$\begin{aligned} \sum_j j\chi_c(N_j^{\max}) &= \chi_c(M) + \chi_c(A_1(f)) + 2\#(A_2^+(f)), \\ \sum_j j\chi_c(N_j^{\min}) &= \chi_c(M) - \chi_c(A_1(f)) - 2\#(A_2^-(f)), \end{aligned}$$

and thus:

$$\begin{aligned} \sum_j \frac{j}{2} [\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})] &= \chi_c(M) + \#(A_2^+(f)) - \#(A_2^-(f)), \\ \sum_j \frac{j}{2} [\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})] &= \chi_c(A_1(f)) + \#(A_2(f)) = \chi_c(\Sigma(f)). \end{aligned}$$

When $\dim N = 3$, we have:

$$\begin{aligned} \sum_j j\chi_c(N_j^{\max}) &= \chi_c(M) + \chi_c(A_1(f)) + 2\chi_c(A_2^+(f)) + \#(A_3(f)) + 2\#(A_3^+(f)), \\ \sum_j j\chi_c(N_j^{\min}) &= \chi_c(M) - \chi_c(A_1(f)) - 2\chi_c(A_2^-(f)) - \#(A_3(f)) - 2\#(A_3^-(f)), \end{aligned}$$

and thus:

$$\sum_j \frac{j}{2} [\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})] = \chi_c(M) + \chi_c(A_2^+(f)) - \chi_c(A_2^-(f)) + \#(A_3^+(f)) - \#(A_3^-(f)),$$

$$\sum_j \frac{j}{2} [\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})] = \chi_c(A_1(f)) + \chi_c(A_2(f)) + 2\#(A_3(f)) = \chi_c(\Sigma(f)) + \#(A_3(f)).$$

5.3. Case $m - n = 0$. Here we assume that M and N are oriented and have the same dimension n . If a point x in M is of type ν , we denote by d_ν the local topological degree of the map-germ $f : (M, x) \rightarrow (N, f(x))$. We assume that f is finite-to-one and that f is locally trivial at infinity. In this situation, it is possible to define the mapping degree of f as follows:

$$\deg f = \sum_{x \in f^{-1}(y)} \deg(f : (M, x) \rightarrow (N, f(x))),$$

where y is a regular value of f .

Theorem 5.11. *Assume that a map $f : M \rightarrow N$ is finite-to-one, locally trivial at infinity and has finitely many singularity types. We also assume that M and N are oriented and that N is connected. Then:*

$$\sum_{\nu} d_{\sigma} \chi_c(\nu(f)) = (\deg f) \chi_c(N),$$

provided that the $\chi_c(\nu(f))$'s are finite.

Proof. We consider the stratification of f defined by the types of singularities (see Nakai's paper [12, §1]) and we define $\mathcal{S}(M)$, $\mathcal{S}(N)$ as the subset algebras generated by the strata and fibers of f . Then $(\mathcal{S}(X), \mathcal{S}(Y))$ fits to the map f . Set $\mu_X = \chi_c$, $\mu_Y = \chi_c$ and:

$$\varphi(x) = \deg(f : (M, x) \rightarrow (N, f(x))).$$

Applying Corollary 2.3 for φ and remarking that $f_*\varphi(y) = \deg f$, we obtain the result. \square

If x is a point of type A_k with k even, we say that x belongs to $A_k^+(f)$ (resp. $A_k^-(f)$) if $\deg\{f : (M, x) \rightarrow (N, f(x))\} = 1$ (resp. -1).

Corollary 5.12. *Assume that f satisfies the assumptions of Theorem 5.11 and has at worst A_n singularities. Then we have:*

$$\sum_{k: \text{even}} [\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))] = (\deg f) \chi_c(N).$$

Proof. Apply the previous theorem and the fact that $\deg\{f : (M, x) \rightarrow (N, f(x))\} = 0$ if $x \in A_k(f)$, k odd. \square

The map $f : (\mathbb{R}^4, 0) \rightarrow (\mathbb{R}^4, 0)$ is an $I_{2,2}^\pm$ singularity if f is defined by:

$$(x, y, a, b) \mapsto (x^2 \pm y^2 + ax + by, xy, a, b).$$

This is the only singularity of stable-germs which is not a Morin singularity from \mathbb{R}^4 to \mathbb{R}^4 . We can state:

Corollary 5.13. *Assume that f satisfies the assumptions of Theorem 5.11 and that $n = 4$. Then we have:*

$$\sum_{k: \text{even}} [\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))] + 2\#(I_{2,2}^-(f)) = (\deg f) \chi_c(N).$$

Proof. Remark that the mapping degree of f_x is 2 (resp. 0) when x is an $I_{2,2}^-$ ($I_{2,2}^+$) point. \square

A similar discussion shows the following:

Theorem 5.14. *Assume that a map $f : M \rightarrow N$ is finite-to-one, locally trivial at infinity and has finitely many singularity types. We assume that M or N may not be orientable and that N is connected. Then:*

$$\sum_{\sigma} d_{\sigma} \chi_c(M_{\sigma}(f)) \equiv (\deg f) \chi_c(N) \pmod{2}.$$

6. Applications to Morin maps

In this section, we apply the results of the previous section to Morin maps. We will consider three different settings : Morin maps from a compact manifold M to a connected manifold N such that $\dim M - \dim N$ is odd, Morin maps from a compact manifold M to a connected manifold N with $\dim M = \dim N$, Morin perturbations of smooth map-germs.

6.1. Morin maps from M^m to N^n , $m - n$ odd. Let $f : M^m \rightarrow N^n$ be a Morin mapping from a compact m -dimensional manifold M to a connected n -dimensional manifold N .

Let us recall that a point p in M is of type A_k if its genotype is x^{k+1} . This means that there exist a local coordinate sytem (x_1, \dots, x_m) centered at p and a local coordinate system (y_1, \dots, y_n) centered at $f(p)$ such that f has the following normal form:

$$\begin{aligned} y_i \circ f &= x_i \text{ for } i \leq n-1, \\ y_n \circ f &= x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i} + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2. \end{aligned}$$

Note that x belongs to $A_k^+(f)$ (resp. $A_k^-(f)$) if and only if $m - n - \lambda + 1$ is even (resp. odd). We should remark also that if k is odd then $x \in A_k^+(f)$ (resp. $A_k^-(f)$) if and only if $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = 0$ (resp. 2) where y' is a regular value of f close to $f(x)$. It is well known that for $k \geq 1$, the $A_k(f)$'s are smooth manifolds of dimension $n - k$, that the $\overline{A_k(f)}$'s are smooth manifolds with boundary and that:

$$\overline{A_k(f)} = \cup_{i \geq k} A_i(f), \quad \partial \overline{A_k(f)} = \cup_{i > k} A_i(f).$$

We will describe more precisely the structure of the $A_k^\pm(f)$'s.

Proposition 6.1. *If k is odd then $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ are compact manifolds with boundary of dimension $n - k$. Furthermore $\partial \overline{A_k^+(f)} = \partial \overline{A_k^-(f)} = \overline{A_{k+1}(f)}$.*

Proof. Let p be a point in $A_k(f)$, k odd. There exist local coordinates around p and $f(p)$ such that f has the form:

$$\begin{aligned} y_i \circ f &= x_i \text{ for } i \leq n-1, \\ y_n \circ f &= x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i} + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2. \end{aligned}$$

Let us write $g = y_n \circ f$. Around p , $A_k(f)$ is defined by $\frac{\partial g}{\partial x_n} = \dots = \frac{\partial^k g}{\partial x_n^k} = 0$ and $x_{n+1} = \dots = x_m = 0$. It is easy to see that this is equivalent to $x_1 = \dots = x_{k-1} = 0$ and $x_n = \dots = x_m = 0$. This proves that $A_k(f)$ is a manifold of dimension $n - k$. Let $q = (q_1, \dots, q_m) \in A_k(f)$ be a point close to p . We have $q_1 = \dots = q_{k-1} = 0$ and $q_n = \dots = q_m = 0$. For $i \in \{k, \dots, n-1\}$, let us put $z_i = x_i - q_i$ and $w_i = y_i - q_i$. For $i \notin \{k, \dots, n-1\}$, let us put $z_i = x_i$ and $w_i = y_i$. Then (z_1, \dots, z_m) and (w_1, \dots, w_n) are local coordinate systems centered at q and $f(q)$. In these systems, f has the form:

$$\begin{aligned} w_i \circ f &= z_i \text{ for } i \leq n-1, \\ w_n \circ f &= z_n^{k+1} + \sum_{i=1}^{k-1} z_i z_n^{k-i} + z_{n+1}^2 + \dots + z_{n+\lambda-1}^2 - z_{n+\lambda}^2 - \dots - z_m^2. \end{aligned}$$

We conclude that q belongs to $A_k^+(f)$ (resp. $A_k^-(f)$) if and only if p belongs to $A_k^+(f)$ (resp. $A_k^-(f)$). This proves that the sets $A_k^+(f)$ and $A_k^-(f)$ are open subsets of $A_k(f)$, hence manifolds of dimension $n - k$.

We know that $\overline{A_k(f)} = \cup_{l \geq k} A_l(f)$. Let $l > k$ and let $p \in A_l(f)$. There are local coordinates systems around p and $f(p)$ such that f has the form:

$$\begin{aligned} y_i \circ f &= x_i \text{ for } i \leq n-1, \\ y_n \circ f &= x_n^{l+1} + \sum_{i=1}^{l-1} x_i x_n^{l-i} + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2. \end{aligned}$$

Let us denote by g the function $y_n \circ f$. We have:

$$A_k(f) = \left\{ \frac{\partial g}{\partial x_n} = \dots = \frac{\partial^k g}{\partial x_n^k} = 0, x_{n+1} = \dots = x_m = 0, \frac{\partial^{k+1} g}{\partial x_n^{k+1}} \neq 0 \right\},$$

and

$$\overline{A_{k+1}(f)} = \left\{ \frac{\partial g}{\partial x_n} = \dots = \frac{\partial^{k+1} g}{\partial x_n^{k+1}} = 0, x_{n+1} = \dots = x_m = 0 \right\}.$$

Let $q = (q_1, \dots, q_n, 0, \dots, 0)$ be a point in $A_k(f)$ close to p . Let us find when $q \in A_k^+(f)$ or $q \in A_k^-(f)$. For this we have to compute $\varphi(q) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(q)})$ where y' is a regular value of f close to $f(q)$. Since it does not depend on the choice of the regular value because $m - n$ is odd, let us compute $\chi(f^{-1}(\tilde{y}) \cap \overline{B_\varepsilon(q)})$ where $\tilde{y} = (q_1, \dots, q_{n-1}, q_n + \epsilon)$ and ϵ is a small real number. So we have to look for the zeros lying close to q of the following system:

$$\begin{cases} y_i \circ f(x) = q_i \text{ for } i \leq n-1 \\ g(x) = g(q) + \epsilon. \end{cases}$$

This system is equivalent to:

$$\begin{cases} x_i = q_i \text{ for } i \leq n-1 \\ g(q_1, \dots, q_{n-1}, q_n + x'_n, x_{n+1}, \dots, x_m) = g(q) + \epsilon. \end{cases}$$

But we have:

$$\begin{aligned} & g(q_1, \dots, q_{n-1}, q_n + x'_n, x_{n+1}, \dots, x_m) \\ &= g(q_1, \dots, q_{n-1}, q_n + x'_n, 0, \dots, 0) + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2 \\ &= g(q) + \sum_{i \geq k+1} \frac{1}{i!} \frac{\partial^i g}{\partial x_n^i}(q) x_n'^i + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2 \\ &= g(q) + g'(x'_n, x_{n+1}, \dots, x_m). \end{aligned}$$

Hence by Khimshiashvili's formula [9], we have : $\varphi(q) = 1 - \deg_0 \nabla g'$, where $\deg_0 \nabla g'$ is the topological degree of the map $\frac{\nabla g'}{\|\nabla g'\|} : S_\varepsilon^{m-n} \rightarrow S^{m-n}$. Two cases are possible. If λ is even then:

$$q \in A_k^+(f) \Leftrightarrow \frac{\partial^{k+1} g}{\partial x_n^{k+1}}(q) > 0 \text{ and } q \in A_k^-(f) \Leftrightarrow \frac{\partial^{k+1} g}{\partial x_n^{k+1}}(q) < 0.$$

If λ is odd then:

$$q \in A_k^+(f) \Leftrightarrow \frac{\partial^{k+1} g}{\partial x_n^{k+1}}(q) < 0 \text{ and } q \in A_k^-(f) \Leftrightarrow \frac{\partial^{k+1} g}{\partial x_n^{k+1}}(q) > 0.$$

Finally we see that the sets $A_k^+(f)$ and $A_k^-(f)$ are in correspondence with the sets $A_k(f) \cap \{\frac{\partial^{k+1} g}{\partial x_n^{k+1}} > 0\}$ and $A_k(f) \cap \{\frac{\partial^{k+1} g}{\partial x_n^{k+1}} < 0\}$, which enables us to conclude. \square

We can state our main theorem which is a slight improvement of a result of T. Fukuda [5] for $N = \mathbb{R}^n$ and O. Saeki [14] for a general N .

Theorem 6.2. *Let $f : M^m \rightarrow N^n$ be a Morin mapping. Assume that M is compact, N is connected and $m - n$ is odd. Then we have:*

$$\chi(M) = \sum_{k: \text{odd}} \left[\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)}) \right].$$

Proof. Applying Corollary 5.4, we get:

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_{k: \text{odd}} [\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))],$$

where χ_f is the Euler characteristic of a regular fiber of f . In this situation, $\chi_f = 0$ because the regular fiber of f is a compact odd-dimensional manifold. Then we remark that $\chi_c(M) = \chi(M)$ because M is compact. Moreover by the additivity of the Euler-Poincaré with compact support, we have:

$$\chi(\overline{A_k^+(f)}) = \chi_c(\overline{A_k^+(f)}) = \chi_c(A_k^+(f)) + \chi_c(\partial(\overline{A_k^+(f)})) = \chi_c(A_k^+(f)) + \chi_c(\overline{A_{k+1}(f)}),$$

$$\chi(\overline{A_k^-(f)}) = \chi_c(\overline{A_k^-(f)}) = \chi_c(A_k^-(f)) + \chi_c(\partial(\overline{A_k^-(f)})) = \chi_c(A_k^-(f)) + \chi_c(\overline{A_{k+1}(f)}).$$

This implies that $\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)}) = \chi_c(A_k^+(f)) - \chi_c(A_k^-(f))$. \square

We end this subsection with two remarks:

- (1) If m is odd then n is even and $\chi(M) = 0$. If k is odd, the dimension of $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ is odd. Furthermore, we have:

$$\chi(\overline{A_k^+(f)}) = \frac{1}{2}\chi(\partial\overline{A_k^+(f)}) = \frac{1}{2}\chi(\overline{A_{k+1}(f)}) = \frac{1}{2}\chi(\partial\overline{A_k^-(f)}) = \chi(\overline{A_k^-(f)}),$$

and $\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)}) = 0$. In this case, our theorem is trivial.

- (2) If m is even and $n = 1$, then we can apply our theorem. In this situation, there is only a finite number of singular points, which are the elements of $A_1^+(f)$ and of $A_1^-(f)$. Theorem 6.2 gives that $\chi(M) = \#A_1^+(f) - \#A_1^-(f)$. We recover the well-known Morse equalities.

6.2. Morin maps from M^n to N^n . Let $f : M^n \rightarrow N^n$ be a Morin mapping from a compact oriented manifold M of dimension n to a connected manifold N of the same dimension. For any $p \in M$, let $\varphi(p)$ be the local topological degree of the map-germ $f : (M, p) \rightarrow (N, f(p))$. Recall that $\varphi(p) = 0$ if $p \in A_k(f)$ and k odd and that $|\varphi(p)| = 1$ if $p \in A_k(f)$ and k even. Hence, if k is even, $A_k(f)$ splits into two subsets $A_k^+(f)$ and $A_k^-(f)$ where $A_k^+(f)$ (resp. $A_k^-(f)$) consists of the points p such that $\varphi(p) = 1$ (resp. $\varphi(p) = -1$). It is well known that the $A_k(f)$'s are smooth manifolds of dimension $n - k$, that the $\overline{A_k(f)}$'s are smooth manifolds with boundary and that:

$$\overline{A_k(f)} = \cup_{i \geq k} A_i(f), \quad \partial\overline{A_k(f)} = \cup_{i > k} A_i(f).$$

Remark that $A_0(f)$ is the set of regular points of f . Let us describe more precisely the structure of the sets $A_k^\pm(f)$.

Proposition 6.3. *If k is even, then $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ are manifolds with boundary of dimension $n - k$ and $\partial\overline{A_k^+(f)} = \partial\overline{A_k^-(f)} = \overline{A_{k+1}(f)}$.*

Proof. Let p be a point in $A_k(f)$, k even. In local coordinates, f is given by:

$$\begin{aligned} y_i \circ f &= x_i \text{ for } i \leq n-1, \\ y_n \circ f &= x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i}. \end{aligned}$$

If we suppose that (x_1, \dots, x_n) and (y_1, \dots, y_n) are coordinates in direct basis, then f has two possible forms:

$$\begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i}, \end{cases} \quad \text{or} \quad \begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = -x_n^{k+1} + \sum_{i=1}^{k-1} (-1)^{k-i} x_i x_n^{k-i}. \end{cases}$$

In the first case, $\varphi(p) = 1$ and in the second case $\varphi(p) = -1$.

We can prove the fact that $A_k^+(f)$ and $A_k^-(f)$ are manifolds of dimension $n - k$ with the same method as in Proposition 6.1. Now let $l > k$ and let $p \in A_l(f)$. Locally f is given by:

$$\begin{aligned} y_i \circ f &= x_i \text{ for } i \leq n-1, \\ y_n \circ f &= \pm x_n^{l+1} + \sum_{i=1}^{l-1} \pm x_i x_n^{l-i}. \end{aligned}$$

Let us denote by g the function $y_n \circ f$. We have:

$$A_k(f) = \left\{ \frac{\partial g}{\partial x_n} = \dots = \frac{\partial^k g}{\partial x_n^k} = 0, \frac{\partial^{k+1} g}{\partial x_n^{k+1}} \neq 0 \right\},$$

and:

$$\overline{A_{k+1}(f)} = \left\{ \frac{\partial g}{\partial x_n} = \dots = \frac{\partial^{k+1} g}{\partial x_n^{k+1}} = 0 \right\}.$$

Let $q = (q_1, \dots, q_n)$ be a point in $A_k(f)$ close to p . Let us find when $q \in A_k^+(f)$ or $q \in A_k^-(f)$. For this we have to compute $\varphi(q)$. Let ϵ be a small real number and let us look for the zeros lying close to q of the following system:

$$\begin{cases} y_i \circ f(x) = q_i \text{ for } i \leq n-1, \\ g(x) = g(q) + \epsilon. \end{cases}$$

This system is equivalent to:

$$\begin{cases} x_i = q_i \text{ for } i \leq n-1, \\ g(q_1, \dots, q_{n-1}, q_n + x'_n) = g(q) + \epsilon. \end{cases}$$

But:

$$g(q_1, \dots, q_{n-1}, q_n + x'_n) = g(q) + \sum_{i \geq k+1} \frac{\partial^i g}{\partial x_n^i}(q) x_n'^i.$$

Then we see that $\varphi(q) = \text{sign} \frac{\partial^{k+1} g}{\partial x_n^{k+1}}(q)$. We conclude as in Proposition 6.1. \square

Theorem 6.4. *Let $f : M^n \rightarrow N^n$ be a Morin mapping. Assume that M is compact and oriented and that N is connected and oriented. We have:*

$$\sum_{k: \text{even}} [\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)})] = (\deg f) \chi(N).$$

This is proved by I. R. Quine [13] when $n = 2$. It appeared in a preprint of I. Nakai [11] for any n .

Proof. By Corollary 5.12, we know that:

$$\sum_{k: \text{even}} [\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))] = (\deg f) \chi_c(N).$$

If N is compact then $\chi_c(N) = \chi(N)$ and if N is not compact then $\deg f = 0$. In both cases the equality $(\deg f) \chi_c(N) = (\deg f) \chi(N)$ is true. With the same arguments as in Theorem 6.2, it is easy to prove that $\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)}) = \chi_c(A_k^+(f)) - \chi_c(A_k^-(f))$. \square

Remark 6.5. When n is odd, $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ are odd-dimensional manifolds with the same boundary and so the left hand-side of the equality vanishes. But the right-hand side is also zero because $\chi(N) = 0$ if N is compact and $\deg f = 0$ if N is not compact. Hence our theorem is trivial in this case.

6.3. Local versions. We give local versions of the global formulas of the previous subsections.

We work first with map-germs $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, $n > p$, which are generic in the sense of Theorem 1' in [4]. There are two cases:

Case I) If the origin 0 is not isolated in $f^{-1}(0)$, i.e $0 \in \overline{f^{-1}(0) \setminus \{0\}}$, then there exist a positive number ε_0 and a strictly increasing function $\delta : [0, \varepsilon_0] \rightarrow [0, +\infty)$ with $\delta(0) = 0$ such that for every ε and δ with $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta < \delta(\varepsilon)$ the following properties hold:

- (1) $f^{-1}(0) \cap S_{\varepsilon}^{n-1}$ is an $(n-p-1)$ -dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S_{\varepsilon_0}^{n-1}$.
- (2) $\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})$ is a smooth manifold with boundary and it is diffeomorphic to $\overline{B_{\varepsilon_0}^n} \cap f^{-1}(S_{\delta(\varepsilon_0)}^{p-1})$.
- (3) $\partial(\overline{B_{\varepsilon}^n} \cap f^{-1}(\overline{B_{\delta}^p}))$ is homeomorphic to S_{ε}^{n-1} .
- (4) The restricted mapping $f : \overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1}) \rightarrow S_{\delta}^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair) and its topological type is independent of ε and δ .

Here B_{ε}^n denotes the open ball of radius ε centered at 0 and S_{ε}^{n-1} the sphere of radius ε centered at 0 in \mathbb{R}^n .

Case II) If the origin 0 is isolated in $f^{-1}(0)$, i.e $0 \notin \overline{f^{-1}(0) \setminus \{0\}}$, then there exists a positive number ε_0 such that for every ε with $0 < \varepsilon \leq \varepsilon_0$ the following properties hold:

- (1) $f^{-1}(S_\varepsilon^{p-1})$ is diffeomorphic to S_ε^{n-1} .
- (2) The restricted mapping $f : f^{-1}(S_\varepsilon^{p-1}) \rightarrow S_\varepsilon^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair) and its topological type is independent of ε .

We will focus first on Case I). Note that in this case, $\overline{B_\varepsilon^n} \cap f^{-1}(\overline{B_\delta^p})$ is a manifold with corners whose topological boundary is the manifold with corners $\overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1}) \cup S_\varepsilon^{n-1} \cap f^{-1}(\overline{B_\delta^p})$. We will use the following notations : $B_{\varepsilon,\delta} = f^{-1}(\overline{B_\delta^p}) \cap \overline{B_\varepsilon^n}$, $\partial B_{\varepsilon,\delta} = \overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1}) \cup S_\varepsilon^{n-1} \cap f^{-1}(\overline{B_\delta^p})$, $C_{\varepsilon,\delta} = B_\varepsilon^n \cap f^{-1}(S_\delta^{p-1})$ and $I_{\varepsilon,\delta}$ is the topological interior of $B_{\varepsilon,\delta}$.

Let us denote by ∂f the restricted mapping $f|_{C_{\varepsilon,\delta}} : C_{\varepsilon,\delta} \rightarrow S_\delta^{p-1}$ and let us assume that it is a Morin mapping. Let us consider a perturbation \tilde{f} of f such that $\tilde{f}|_{I_{\varepsilon,\delta}} : I_{\varepsilon,\delta} \rightarrow B_\delta^p$ is a Morin mapping and $\tilde{f} = f$ in a neighborhood of $C_{\varepsilon,\delta}$.

Our aim is to generalize Theorem 2 of [4] which deals with map-germs from \mathbb{R}^n to \mathbb{R} , i.e to relate the topology of $\text{Lk}(f) = f^{-1}(0) \cap S_\varepsilon^{n-1}$ to the topology of the singular set of \tilde{f} and to the topology of the singular set of ∂f . As in the previous sections, we will denote by $A_k(\tilde{f})$ (resp. $A_k(\partial f)$), the set of singular points of \tilde{f} (resp. ∂f) of type A_k . The first result is a local version of Saeki's formula (Theorem 2.3 in [14]).

Theorem 6.6. *We have:*

$$\psi(\text{Lk}(f)) \equiv 1 + \sum_{k=1}^{p-1} \psi(A_k(\partial f)) + \#A_p(\tilde{f}) \pmod{2},$$

where ψ denotes the semi-characteristic.

Proof. Note that for $\tilde{\delta}$ a sufficiently small regular value of \tilde{f} ($|\tilde{\delta}| \leq \delta$), we have:

$$\chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) \equiv \chi(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) \equiv \chi(\tilde{f}^{-1}(\tilde{\delta}) \cap B_{\varepsilon,\delta}) \equiv \psi(\tilde{f}^{-1}(\tilde{\delta}) \cap S_\varepsilon^{n-1}) \equiv \psi(\text{Lk}(f)) \pmod{2}.$$

The last equality comes from the fact that f has an isolated singularity, that $\tilde{f}^{-1}(\tilde{\delta})$ intersects S_ε^{n-1} transversally and that \tilde{f} is close to f .

On the one hand, applying Theorem 5.1, Theorem 5.6 and their corollaries to the restriction of \tilde{f} to $I_{\varepsilon,\delta}$, we obtain:

$$\sum_{k:\text{even}} \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) \equiv \psi(\text{Lk}(f)) \pmod{2}.$$

On the other hand, by additivity, we have:

$$1 \equiv \chi_c(I_{\varepsilon,\delta}) \equiv \sum_k \chi_c(I_{\varepsilon,\delta} \cap A_k(\tilde{f})) \pmod{2}.$$

For each $k \geq 1$, we have:

$$\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}} = A_k(\tilde{f}) \cap I_{\varepsilon,\delta} \sqcup \overline{A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}} \sqcup \overline{A_k(\tilde{f}) \cap C_{\varepsilon,\delta}},$$

because if ε and δ are small enough the singular set of \tilde{f} does not intersect $f^{-1}(\overline{B_\delta^p}) \cap S_\varepsilon^{n-1}$.

Before carrying on with our computations, let us observe that for $k \in \{1, \dots, p-1\}$, $A_k(\tilde{f}) \cap C_{\varepsilon,\delta} = A_k(\partial f)$. It is not difficult to see this with the characterization of the A_k sets by the ranks of the iterate jacobians. Moreover, using the characterization of the A_k^+ and A_k^- sets by the Euler characteristic of the nearby fiber, we can say that $\overline{A_k^+(\tilde{f}) \cap C_{\varepsilon,\delta}} = A_k^+(\partial f)$ and $\overline{A_k^-(\tilde{f}) \cap C_{\varepsilon,\delta}} = A_k^-(\partial f)$. Hence:

$$\begin{aligned} \chi(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) &\equiv \chi_c(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) \\ &\equiv \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}}) + \chi_c(\overline{A_k(\tilde{f}) \cap C_{\varepsilon,\delta}}) \\ &\equiv \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}}) \pmod{2}, \end{aligned}$$

because $\overline{A_k(\tilde{f})} \cap C_{\varepsilon,\delta}$ is a compact boundary. Furthermore, we have:

$$\chi(A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}) = \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap C_{\varepsilon,\delta}) = \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}).$$

Finally, for each k , $\chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) = \chi(\overline{A_k(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta})$, and so:

$$\begin{aligned} \psi(\text{Lk}(f)) &\equiv 1 + \sum_{k=1}^p \chi(\overline{A_k(\tilde{f})} \cap I_{\varepsilon,\delta}) \pmod{2}, \\ \psi(\text{Lk}(f)) &\equiv 1 + \sum_{k=1}^{p-1} \psi(A_k(\partial f)) + \#A_p(\tilde{f}) \pmod{2}. \end{aligned}$$

□

Let us examine some special cases. When $p = 1$, we find:

$$\psi(\text{Lk}(f)) \equiv 1 + \#A_1(\tilde{f}) \equiv 1 + \deg_0 \nabla f \pmod{2},$$

where $\deg_0 \nabla f$ is the topological degree of the map $\frac{\nabla f}{\|\nabla f\|} : S_{\varepsilon}^{n-1} \rightarrow S^{n-1}$. This due to the fact \tilde{f} is a Morse function and the points in $A_1(\tilde{f})$ are exactly its critical points. When $p = 2$, we find:

$$\psi(\text{Lk}(f)) \equiv 1 + \psi(A_1(\partial f)) + \#A_2(\tilde{f}) \pmod{2}.$$

If \tilde{f} is close to f then $\psi(A_1(\partial f))$ is equal to $\frac{1}{2}b(C(f)) \pmod{2}$ where $C(f)$ denotes critical locus of f and $b(C(f))$ the number of branches of $\bar{C}(f)$. Hence:

$$\psi(\text{Lk}(f)) \equiv 1 + \frac{1}{2}b(C(f)) + \#A_2(\tilde{f}) \pmod{2}.$$

Since $b(C(f))$ is a topological invariant of f , we deduce that $\#A_2(\tilde{f}) \pmod{2}$ is a topological invariant of f . Similarly if $p = 3$, this gives:

$$\psi(\text{Lk}(f)) \equiv 1 + \psi(C(f) \cap \partial B_{\varepsilon,\delta}) + \frac{1}{2}b(C(f|_{C(f)})) + \#A_3(\tilde{f}) \pmod{2}.$$

In the sequel, we will improve Theorem 6.6 in some situations. Let us assume that $n - p$ is odd.

Theorem 6.7. *If $n - p$ is odd, then we have:*

$$\chi(\text{Lk}(f)) = 2 - 2 \sum_{k:\text{odd}} \left[\chi(\overline{A_k^+(\tilde{f})} \cap I_{\varepsilon,\delta}) - \chi(\overline{A_k^-(\tilde{f})} \cap I_{\varepsilon,\delta}) \right].$$

Furthermore, when n is odd and p is even, we have:

$$\chi(\text{Lk}(f)) = 2 - \sum_{k:\text{odd}} \left[\chi(\overline{A_k^+(\partial f)}) - \chi(\overline{A_k^-(\partial f)}) \right].$$

Proof. With the same notations as in Theorem 6.6, we can write:

$$\chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap B_{\varepsilon,\delta}) = \chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) + \chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap \partial B_{\varepsilon,\delta}),$$

thus:

$$\frac{1}{2}\chi(\text{Lk}(f)) = \chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) + \chi(\text{Lk}(f)).$$

Therefore, we get:

$$\chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) = -\frac{1}{2}\chi(\text{Lk}(f)).$$

Applying Corollary 5.4, we obtain:

$$\chi_c(I_{\varepsilon,\delta}) + \frac{1}{2}\chi(\text{Lk}(f))\chi_c(B_{\delta}^p) = \sum_{k:\text{odd}} \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) - \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}).$$

Let us compute $\chi_c(I_{\varepsilon,\delta})$. We have:

$$\chi(B_{\varepsilon,\delta}) = \chi_c(B_{\varepsilon,\delta}) = \chi_c(I_{\varepsilon,\delta}) + \chi_c(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})) + \chi_c(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})).$$

If n is odd and p is even, we have:

$$\chi(B_{\varepsilon,\delta}) = \frac{1}{2}\chi(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) + \frac{1}{2}\chi(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})),$$

and:

$$\begin{aligned} \chi(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})) &= \chi_c(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})) \\ &= \chi_c(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})) + \chi_c(S_{\varepsilon}^{n-1} \cap f^{-1}(S_{\delta}^{p-1})) \\ &= \chi_c(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})) + \chi(S_{\varepsilon}^{n-1} \cap f^{-1}(S_{\delta}^{p-1})) \\ &= \chi_c(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})). \end{aligned}$$

Thus we obtain:

$$\begin{aligned} \chi_c(I_{\varepsilon,\delta}) &= \frac{1}{2}\chi(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) + \frac{1}{2}\chi(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})) - \chi(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})) - \chi(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) \\ &= -\frac{1}{2}\chi(\overline{B_{\varepsilon}^n} \cap f^{-1}(S_{\delta}^{p-1})) - \frac{1}{2}\chi(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) = -\chi(B_{\varepsilon,\delta}) = -1. \end{aligned}$$

Finally we get:

$$\frac{1}{2}\chi(\text{Lk}(f)) = 1 + \sum_{k:\text{odd}} \left[\chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) - \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}) \right],$$

which means:

$$\chi(\text{Lk}(f)) = 2 + 2 \sum_{k:\text{odd}} \left[\chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) - \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}) \right].$$

Since $\dim A_k^+(\tilde{f}) = \dim A_k^-(\tilde{f}) = p - k$ is odd, we can establish using the same arguments as above that:

$$\begin{aligned} \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) &= -\chi(\overline{A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}}) = -\frac{1}{2}\chi(\overline{A_k^+(\tilde{f})} \cap C_{\varepsilon,\delta}) - \frac{1}{2}\chi(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}), \\ \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}) &= -\chi(\overline{A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}}) = -\frac{1}{2}\chi(\overline{A_k^-(\tilde{f})} \cap C_{\varepsilon,\delta}) - \frac{1}{2}\chi(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}). \end{aligned}$$

Finally, we obtain:

$$\chi(\text{Lk}(f)) = 2 - 2 \sum_{k:\text{odd}} \left[\chi(\overline{A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}}) - \chi(\overline{A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}}) \right] = 2 - \sum_{k:\text{odd}} \left[\chi(\overline{A_k^+(\partial f)}) - \chi(\overline{A_k^-(\partial f)}) \right].$$

If n is even and p is odd, then:

$$\begin{aligned} \chi_c(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})) &= -\chi(B_{\varepsilon}^n \cap f^{-1}(S_{\delta}^{p-1})) = -\frac{1}{2}\chi(S_{\varepsilon}^{n-1} \cap f^{-1}(S_{\delta}^{p-1})), \\ \chi_c(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) &= \chi(S_{\varepsilon}^{n-1} \cap f^{-1}(\overline{B_{\delta}^p})) = \frac{1}{2}\chi(S_{\varepsilon}^{n-1} \cap f^{-1}(S_{\delta}^{p-1})). \end{aligned}$$

So $\chi_c(I_{\varepsilon,\delta}) = \chi(B_{\varepsilon,\delta}) = 1$, and:

$$1 - \sum_{k:\text{odd}} \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) + \sum_{k:\text{odd}} \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}) = \frac{1}{2}\chi(\text{Lk}(f)),$$

and then:

$$\chi(\text{Lk}(f)) = 2 - 2 \left(\sum_{k:\text{odd}} \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) - \sum_{k:\text{odd}} \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon,\delta}) \right).$$

Here $\dim A_k^+(\tilde{f}) = \dim A_k^-(\tilde{f}) = p - k$ is even when k is odd. We have:

$$\begin{aligned} \chi(\overline{A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}}) &= \chi_c(\overline{A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}}) \\ &= \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_k^+(\tilde{f})} \cap C_{\varepsilon,\delta}) \end{aligned}$$

$$\begin{aligned}
&= \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon, \delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon, \delta}) + \chi(\overline{A_k^+(\tilde{f})} \cap C_{\varepsilon, \delta}) \\
&= \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon, \delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon, \delta}).
\end{aligned}$$

Hence:

$$\chi(\overline{A_k^+(\tilde{f})} \cap I_{\varepsilon, \delta}) - \chi(\overline{A_k^-(\tilde{f})} \cap I_{\varepsilon, \delta}) = \chi_c(A_k^+(\tilde{f}) \cap I_{\varepsilon, \delta}) - \chi_c(A_k^-(\tilde{f}) \cap I_{\varepsilon, \delta}).$$

□

The same results hold in Case II) replacing $B_\varepsilon^n \cap f^{-1}(S_\varepsilon^{p-1})$ with $f^{-1}(S_\varepsilon^{p-1})$, which is diffeomorphic to S_ε^{n-1} , $B_{\varepsilon, \delta}$ with $f^{-1}(\overline{B_\varepsilon^p})$, $I_{\varepsilon, \delta}$ with the topological interior of $f^{-1}(\overline{B_\varepsilon^p})$ and $\chi(\text{Lk}(f))$ with 0.

Now we work with map-germs from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^n, 0)$. Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a map-germ such that 0 is isolated in $f^{-1}(0)$. We assume that f is generic in the sense of Theorem 3 in [3] : there exists a positive number ε_0 such that for any number ε with $0 < \varepsilon \leq \varepsilon_0$, we have:

- (1) $\tilde{S}_\varepsilon^{n-1} = f^{-1}(S_\varepsilon^{n-1})$ is a homotopy $(n-1)$ -sphere which, if $n \neq 4, 5$, is diffeomorphic to the natural $(n-1)$ -sphere S^{n-1} ,
- (2) the restricted mapping $f|_{\tilde{S}_\varepsilon^{n-1}} : \tilde{S}_\varepsilon^{n-1} \rightarrow S_\varepsilon^{n-1}$ is topological stable (C^∞ stable if (n, p) is a nice pair),
- (3) letting $\tilde{B}_\varepsilon^n = f^{-1}(\overline{B_\varepsilon^n})$, the restricted mapping $f|_{\tilde{B}_\varepsilon^n} : \tilde{B}_\varepsilon^n \setminus \{0\} \rightarrow \overline{B_\varepsilon^n} \setminus \{0\}$ is proper, topologically stable (C^∞ stable if (n, p) is nice) and topologically equivalent (C^∞ equivalent if (n, p) is nice) to the product mapping:

$$(f|_{\tilde{S}_\varepsilon^{n-1}}) \times \text{Id}_{(0, \varepsilon)} : \tilde{S}_\varepsilon^{n-1} \times (0, \varepsilon) \rightarrow S_\varepsilon^{n-1} \times (0, \varepsilon),$$

defined by $(x, t) \mapsto (f(x), t)$,

- (4) consequently, $f|_{\tilde{B}_\varepsilon^n} : \tilde{B}_\varepsilon^n \rightarrow \overline{B_\varepsilon^n}$ is topologically equivalent to the cone:

$$C(f|_{\tilde{S}_\varepsilon^{n-1}}) : \tilde{S}_\varepsilon^{n-1} \times [0, \varepsilon) / \tilde{S}_\varepsilon^{n-1} \times \{0\} \rightarrow S_\varepsilon^{n-1} \times [0, \varepsilon) / S_\varepsilon^{n-1} \times \{0\},$$

of the stable mapping $f|_{\tilde{S}_\varepsilon^{n-1}} : \tilde{S}_\varepsilon^{n-1} \rightarrow S_\varepsilon^{n-1}$ defined by

$$C(f|_{\tilde{S}_\varepsilon^{n-1}})(x, t) = (f(x), t).$$

Note that in this case $\tilde{B}_\varepsilon = f^{-1}(\overline{B_\varepsilon^n})$ is a smooth manifold with boundary $f^{-1}(S_\varepsilon^{n-1})$. This last manifold has the homotopy type of S^{n-1} .

We will keep the notations of the previous sections. We denote by \tilde{B}_ε the set $f^{-1}(\overline{B_\varepsilon^n})$, by \tilde{I}_ε its topological interior and by $\partial\tilde{B}_\varepsilon$ its boundary. We denote by ∂f the restricted mapping $f|_{\partial\tilde{B}_\varepsilon} : \partial\tilde{B}_\varepsilon \rightarrow S_\varepsilon^{n-1}$ and we assume that it is a Morin mapping.

Let us consider a perturbation \tilde{f} of f such that $\tilde{f}|_{\tilde{I}_\varepsilon} : \tilde{I}_\varepsilon \rightarrow B_\varepsilon^n$ is a Morin mapping and $\tilde{f} = f$ in a neighborhood of $\partial\tilde{B}_\varepsilon$.

The main result is a local version of Corollary 5.12.

Theorem 6.8. *We have:*

$$\deg_0 f = \sum_{k: \text{even}} \left[\chi(\overline{A_k^+(\tilde{f})} \cap \tilde{I}_\varepsilon) - \chi(\overline{A_k^-(\tilde{f})} \cap \tilde{I}_\varepsilon) \right],$$

where $\deg_0 f$ is the local topological degree of f at 0.

Proof. Using Corollary 5.12, we obtain:

$$(\deg_0 f)(-1)^n = \sum_{k: \text{even}} \chi_c(A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon) - \chi_c(A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon).$$

It remains to relate the Euler characteristics with compact support to the topological Euler characteristics. But, as in Theorem 6.7, we have:

$$\chi_c(A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon) - \chi_c(A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon) = (-1)^{n-k} \left(\chi(\overline{A_k^+(\tilde{f})} \cap \tilde{I}_\varepsilon) - \chi(\overline{A_k^-(\tilde{f})} \cap \tilde{I}_\varepsilon) \right).$$

□

Corollary 6.9. *If n is odd, we have:*

$$2 \deg_0 f = \sum_{k: \text{even}} \left[\chi(\overline{A_k^+(\partial f)}) - \chi(\overline{A_k^-(\partial f)}) \right].$$

Corollary 6.10.

$$\deg_0 f \equiv 1 + \sum_k \psi(\overline{A_k(\partial f)}) + \#A_n(\tilde{f}) \pmod{2}.$$

Proof. We have:

$$1 = \chi(\tilde{B}_\varepsilon) = \chi(\overline{A_0^+(\tilde{f}) \cap \tilde{I}_\varepsilon}) + \chi(\overline{A_0^-(\tilde{f}) \cap \tilde{I}_\varepsilon}) - \chi(\overline{A_1(\tilde{f}) \cap \tilde{I}_\varepsilon}),$$

hence:

$$\chi(\overline{A_0^+(\tilde{f}) \cap \tilde{I}_\varepsilon}) - \chi(\overline{A_0^-(\tilde{f}) \cap \tilde{I}_\varepsilon}) \equiv 1 + \chi(\overline{A_1(\tilde{f}) \cap \tilde{I}_\varepsilon}) \equiv 1 + \psi(\overline{A_1(\partial f)}) \pmod{2}.$$

Similarly, if k is even and $\dim A_k > 0$, then:

$$\chi(\overline{A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon}) - \chi(\overline{A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon}) \equiv \chi(\overline{A_k(\tilde{f}) \cap \tilde{I}_\varepsilon}) + \chi(\overline{A_{k+1}(\tilde{f}) \cap \tilde{I}_\varepsilon}) \pmod{2}.$$

Thus we obtain that:

$$\chi(\overline{A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon}) - \chi(\overline{A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon}) \equiv \begin{cases} \psi(\overline{A_k(\partial f)}) + \psi(\overline{A_{k+1}(\partial f)}) & \text{if } \dim A_k(\tilde{f}) > 1 \\ \psi(\overline{A_k(\partial f)}) + \#A_{k+1}(\tilde{f}) & \text{if } \dim A_k(\tilde{f}) = 1 \\ \#A_k(\tilde{f}) & \text{if } \dim A_k(\tilde{f}) = 0 \end{cases}$$

modulo 2. □

If $n = 2$, this gives:

$$\deg_0 f \equiv 1 + \frac{1}{2}b(C(f)) + \#A_2(\tilde{f}) \pmod{2},$$

and we recover Theorem 2.1 of T. Fukuda and G. Ishikawa [6].

If $n = 3$, this gives:

$$\deg_0 f \equiv 1 + \psi(C(f) \cap \partial \tilde{B}_\varepsilon) + \frac{1}{2}b(C(f|_{C(f)})) + \#A_3(\tilde{f}) \pmod{2}.$$

7. Complex maps

We end with some remarks in the complex case. Let $f : M \rightarrow N$ be a holomorphic map between complex manifolds M and N . We assume that N is connected. We assume that f is locally infinitesimally stable in J. Mather's sense.

Let c_σ denote the Euler characteristic of the local generic fiber of the map-germ of singular type σ . Let χ_f denote the Euler characteristics of the generic fibers of f .

Theorem 7.1. *If a locally infinitesimally stable map $f : M \rightarrow N$ does not have singularities at infinity, then*

$$\sum_{\sigma} c_{\sigma} \chi_c(M_{\sigma}(f)) = \chi_f \chi_c(N).$$

Proof. Apply Corollary 2.4. □

Corollary 7.2. *If a Morin map $f : M \rightarrow N$ is locally trivial at infinity, then:*

$$\chi_c(M) + (-1)^{m-n} \sum_{k=1}^n \chi_c(\overline{A_k(f)}) = \chi_f \chi_c(N)$$

where m denotes the complex dimension of M and n denotes the complex dimension of N .

We should remark that this formula was firstly formulated by Y. Yomdin (see [19]). Note also that when $m = n$, then χ_f is also the topological degree of f .

Let $f = (f_1, f_2) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a holomorphic map-germ with $c(f) < \infty$ where:

$$c(f) = \dim_{\mathbb{C}} \mathcal{O}_2 \Big/ I_2 \left(\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial J}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \frac{\partial J}{\partial x_2} \end{array} \right), \quad J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}.$$

Corollary 7.3 ([8, (1.8)]). *Let $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be holomorphic map-germs with $c(f) < \infty$, $c(g) < \infty$. Let f_t, g_t denote stable perturbations of f, g . If f and g are topologically right-left equivalent, then $\#A_2(f_t) = \#A_2(g_t)$.*

Proof. Since the critical set can be characterized topologically, $(\mathbb{C}^2, \Sigma(f), 0)$ and $(\mathbb{C}^2, \Sigma(g), 0)$ are topologically equivalent, and they have the same Milnor number. Thus their smoothings have the same Euler characteristic and $\chi(\overline{A_1(f_t)}) = \chi(\overline{A_1(g_t)})$. By Corollary 7.2, we have:

$$\begin{aligned} 1 + \chi_c(\overline{A_1(f_t)}) + \#A_2(f_t) &= \deg_0 f, \\ 1 + \chi_c(\overline{A_1(g_t)}) + \#A_2(g_t) &= \deg_0 g, \end{aligned}$$

and, since $\deg_0 f = \deg_0 g$, we conclude the result. \square

Remark 7.4. Consider the map germ $f = (f_1, f_2) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)$, $n > 2$. Take a stable perturbation f_t of f . We have:

$$(7.1) \quad \chi_c(\overline{A_1(f_t)}) + \#A_2(f_t) = (-1)^n(\chi_f - 1).$$

Consider the map $F : (\mathbb{C}^n, 0) \times (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ defined by $F(x, t) = (f_t(x), t)$. Since $\overline{A_1(F)}$ is determinantal, it is Cohen-Macaulay. So the map $\overline{A_1(F)} \rightarrow (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat. So $\overline{A_1(f_t)}$ is a smoothing of $\overline{A_1(f)}$ and its Euler characteristic is described by the Milnor number of $\overline{A_1(f)}$: $\chi_c(\overline{A_1(f_t)}) = 1 - \mu(\overline{A_1(f)})$, and we conclude that $\mu(\overline{A_1(f)})$ and χ_f determine $\#(A_2(f_t))$. Now we assume that f is \mathcal{A} -finite. Then, we have:

$$\begin{aligned} 1 - \mu(\overline{A_1(f)}) &= \chi_c(\overline{A_1(f_t)}) = \chi_c(f_t(\overline{A_1(f_t)})) + d(f_t) \\ &= 1 - \mu(f(\overline{A_1(f)})) + 2\#(A_2(f_t)) + 2d(f_t), \end{aligned}$$

where $d(f_t)$ denotes the number of double fold $(A_{1,1})$ points of f_t nearby 0. Combining this with (7.1), we obtain:

$$3\#A_2(f_t) + 2d(f_t) = \mu(f(\overline{A_1(f)})) - 1 + (-1)^n(\chi_f - 1).$$

We conclude that $3\#A_2(f_t) + 2d(f_t)$ (and thus $\#A_2(f_t) \bmod 2$) is a topological invariant of f .

Remark 7.5. Consider a map germ $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$, $x \mapsto y = f(x)$. Take a stable perturbation f_t of f . Then we obtain:

$$1 + \chi_c(\overline{A_1(f_t)}) + \chi_c(\overline{A_2(f_t)}) + \#A_3(f_t) = \deg f.$$

Consider the map $F : (\mathbb{C}^3, 0) \times (\mathbb{C}, 0) \rightarrow (\mathbb{C}^3, 0) \times (\mathbb{C}, 0)$ defined by $F(x, t) = (f_t(x), t)$. Since $\overline{A_2(F)}$ is determinantal, it is Cohen-Macaulay. We obtain that the map $\overline{A_2(F)} \rightarrow (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat. So $\overline{A_2(f_t)}$ is a smoothing of $\overline{A_2(f)}$ and its Euler characteristic $\chi_c(\overline{A_2(f_t)})$ is described by Milnor number of $\overline{A_2(f)}$ when $\overline{A_2(f)}$ has an isolated singularity at 0. This means $\#(A_3(f_t))$ is determined by $\mu(\overline{A_1(f)})$, $\mu(\overline{A_2(f)})$ and $\deg f$:

$$\#(A_3(f_t)) = \deg f - \mu(\overline{A_1(f)}) + \mu(\overline{A_2(f)}) - 3,$$

when $\overline{A_1(f)}$ and $\overline{A_2(f)}$ have isolated singularities at 0.

Remark 7.6. Consider a map germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^3, 0)$, $n > 3$. Take a stable perturbation f_t of f . Then we obtain:

$$\chi_c(\overline{A_1(f_t)}) + \chi_c(\overline{A_2(f_t)}) + \#A_3(f_t) = (-1)^n(1 - \chi_f).$$

Consider the map $F : (\mathbb{C}^n, 0) \times (\mathbb{C}, 0) \rightarrow (\mathbb{C}^3, 0) \times (\mathbb{C}, 0)$ defined by $F(x, t) = (f_t(x), t)$. Since $\overline{A_1(F)}$ is determinantal, it is Cohen-Macaulay. We obtain that the map $\overline{A_1(F)} \rightarrow (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat, and $\overline{A_1(f_t)}$ is a smoothing, which is determinantal. So the topology of $\overline{A_1(f_t)}$ is determined by $\overline{A_1(f)}$ when $\overline{A_1(f)}$ has isolated singularity at 0. By Theorem 2.9 in [7], $\overline{A_2(F)}$ is Cohen-Macaulay if and only if $n = 4, 5$. We thus obtain that the map $\overline{A_2(F)} \rightarrow (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat, if only if $n = 4, 5$. Assume that $n = 4, 5$. Then $\overline{A_2(f_t)}$ is a smoothing of $\overline{A_2(f)}$ and its Euler characteristic $\chi_c(\overline{A_2(f_t)})$ is described by the Milnor number of $\overline{A_2(f)}$: $\chi_c(\overline{A_2(f_t)}) = 1 - \mu(\overline{A_2(f)})$. This means $\#(A_3(f_t))$ is determined by $\mu(\overline{A_1(f)})$, $\mu(\overline{A_2(f)})$ and χ_f . When $n \geq 6$, we do not know whether $\chi_c(\overline{A_2(f_t)}) = 1 - \mu(\overline{A_2(f)})$ holds or not.

The following example also shows that the reduced structure of singularities locus may not fit the context of deformation of maps.

Example 7.7. Let us consider the image of the map $g : \mathbb{C} \rightarrow \mathbb{C}$ defined by $s \mapsto (s^3, s^4, s^5)$, which Milnor number μ is 4. The defining ideal is:

$$I_0 = \langle xz - y^2, yz - x^3, x^2y - z^2 \rangle.$$

We know it defines a Cohen-Macaulay space. Consider the map:

$$G : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^4, 0) \text{ defined by } (s, t) \mapsto (x, y, z, t) = (st + s^3, s^4, s^5, t).$$

Remark that $g_0(s) = g(s)$ where $G(s, t) = (g_t(s), t)$. The image of g_t , $t \neq 0$, is nonsingular, and its Euler characteristic is 1, which is not $1 - \mu$. Let us see what happens in this example. Eliminating s from the ideal generated by:

$$x - st - s^3, y - s^4, z - s^5,$$

we obtain the ideal:

$$I = \langle z^2 - x^2y + ty^2 + txz, xy^2 - x^2z + tyz + t^2xy - t^3z, y^3 - xyz + tx^2y - t^2y^2 + t^2xz, \\ xyz + tz^2 - x^4 + 2tx^2y + 2t^2xz + t^4y, y^2z - xz^2 + tx^2z - 2t^2yz - t^3xy + t^4z \rangle$$

of $\mathbb{C}\{x, y, z, t\}$. We remark that the variety X defined by the ideal I is not Cohen-Macaulay. We also remark that this defines a reduced space, but the fiber $\pi^{-1}(0)$, where $\pi : X \rightarrow \mathbb{C}$ is the projection $\pi(x, y, z, t) = t$, is not reduced, since

$$\mathbb{C}\{x, y, z, t\}/I \otimes_{\mathbb{C}} \mathbb{C}\{t\}/\langle t \rangle \simeq \mathbb{C}\{x, y, z\}/I_0 \cap \langle x, y^3, y^2z, z^2 \rangle.$$

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